Short-Time Bohmian Trajectories

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- The latter is used to prove that **short-time Bohmian trajectories are classical Hamiltonian trajectories;**
- The possibility of using this result to prove a quantum Zeno effect is shortly discussed.

References

M. de Gosson and B. Hiley. Short-time quantum propagator and Bohmian trajectories. Physics Letters A. Available online 10 September 2013.

M. de Gosson and B. Hiley. Zeno Paradox for Bohmian Trajectories: The Unfolding of the Metatron. arXiv:1010.2622v3 [quant-ph]; Submitted.

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Quantum motion

Consider a Hamiltonian function $H = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + V(x,t)$ and let Ψ be a solution of the associated Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}\Psi \quad , \quad \widehat{H} = H(x, -i\hbar\nabla_x).$$

Writing $\Psi = Re^{iS/\hbar}$, $R > 0$, and setting $\rho = R^2$ insertion in that equation yields a continuity equation and a Hamilton–Jacobi equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$

$$\frac{\partial S}{\partial t} = H(x, -i\hbar\nabla_x S) + Q^{\Psi}.$$

Quantum motion

In the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$

the velocity field is given by

$$v = (\frac{1}{m_1}\frac{\partial S}{\partial x_1}, ..., \frac{1}{m_n}\frac{\partial S}{\partial x_n})$$

and in the Hamilton–Jacobi equation the term $Q^{\Psi} = Q^{\Psi}(x, t)$ ("quantum potential") is given by

$$Q^{\Psi} = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \frac{1}{R} \frac{\partial^2 R}{\partial x_j^2}.$$

We define: $H^{\Psi} = H + Q^{\Psi}$. Notice that H^{Ψ} is generally time-dependent, even if H isn't.

Quantum motion

Following the theory of quantum motion ("Bohmian mechanics") the (system of) particle(s) is guided by the wavefunction Ψ and follows a Hamiltonian trajectory: $t \longmapsto (x^{\Psi}(t), p^{\Psi}(t))$ where x^{Ψ} and p^{Ψ} are solutions of the system

$$\begin{aligned} \dot{x}^{\Psi} &= \nabla_x H^{\Psi}(x^{\Psi}, p^{\Psi}, t) \\ \dot{p}^{\Psi} &= -\nabla_p H^{\Psi}(x^{\Psi}, p^{\Psi}, t) \end{aligned}$$

with initial conditions

$$x^{\Psi}(0) = x_0 \quad , \quad p^{\Psi}(0) = \nabla_x S(x, 0).$$

Equivalently:

$$\dot{x}_j^{\Psi} = \frac{\hbar}{m_j} \operatorname{Im} \frac{1}{\Psi} \frac{\partial \Psi}{\partial x_j}.$$

Global existence and uniqueness for the Bohmian dynamics has been proven by Berndl, *et al.* (1995).

An elementary example

Choose for initial wavefunction a centered Gaussian

$$\psi_0(x) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp\left[-\frac{x^2}{4\sigma_0^2} + ikx\right]$$

and take $H = p^2/2m$ (the free motion Hamiltonian). Then

$$\psi(x,t) = \frac{1}{(2\pi A_t^2)^{1/4}} \exp\left[-\frac{(x-v_g)^2}{4\sigma_0 A_t} + ik\left(x - \frac{v_g t}{2}\right)\right]$$

where $v_g = \hbar k/m$ is the group velocity and $A_t = \sigma_0(1 + i\hbar t/2m\sigma_0^2)$. After a few calculations we get

$$x^\psi(t) = x_0\left(1 + \frac{\hbar^2 t^2}{4m^2\sigma_0^4}\right)^{1/2} + v_g t;$$

for $t \longrightarrow 0$ we thus have

$$x^\psi(t) = \underbrace{x_0 + v_g t}_{\text{Classical motion}} + O(t^2).$$

Quantum motion

Consider the case of a point source located at x_0. We model it by requiring that the wavefunction satisfies

$$i\hbar \frac{\partial \Psi}{\partial t} = H(x, -i\hbar \nabla_x)\Psi$$

together with the initial condition

$$\lim_{t \to 0} \Psi(x, t) = \delta(x - x_0).$$

The function Ψ is thus here the quantum propagator $G(x, x_0, t)$ (Green function) determined by the Hamiltonian: that is every solution of Schrödinger's equation with initial datum $\Psi(x, 0) = \Psi_0(x)$ can be written

$$\Psi(x, t) = \int G(x, x_0, t)\Psi_0(x_0)d^n x_0.$$

Generating function, action, eikonal

For the free particle ($V = 0$) in one dimension the propagator is

$$G(x, x_0, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{i}{\hbar} m \frac{(x - x_0)^2}{2t}\right].$$

The function

$$S(x, x_0; t) = m\frac{(x - x_0)^2}{2t}$$

is a generating function for the motion: the equations

$$p = \frac{\partial S}{\partial x}(x, x_0; t) \quad , \quad p_0 = -\frac{\partial S}{\partial x_0}(x, x_0; t)$$

are equivalent to the equations of motion

$$x = x_0 + \frac{p_0}{m}t \quad , \quad p = p_0.$$

The function $S(x, x_0; t)$ is the **action** needed to go from x_0 to x in time t with velocity p_0/m. It is called **eikonal** in geometric optics ($=$ optical path).

Generating function, action, eikonal

In Quantum Physics (e.g. the theory of Feynman integral) one looks for short-time approximations to the propagator; one first tries

$$G(x,x_0,t)=\left(\frac{1}{2\pi i\hbar}\right)^{n/2}\sqrt{\rho(x,x_0,t)}\exp\left[\frac{i}{\hbar}S(x,x_0;t)\right]$$

where $S(x,x_0;t)$ is the generating function for the flow determined by $H=\sum_{j=1}^{n}\frac{p_j^2}{2m_j}+V(x,t)$ and

$$\rho(x,x_0,t)=\det\left[-S''_{xx_0}(x,x_0,t)\right]$$

is the Van Vleck determinant (= density of trajectories).
But $S(x,x_0;t)$ is not easy to calculate explicitly, so one looks for **approximations**.

Surely you're joking, Mr. Feynman!

- The literature abounds with approximations for $t \longrightarrow 0$ to the generating function. Among the most popular, the "mid-point rules"

$$
\begin{aligned}
S(x, x_0; t) &\approx \sum_{j=1}^{n} m_j \frac{(x_j - x_{0,j})^2}{2t} - V(\tfrac{1}{2}(x + x_0))t \\
&\text{or} \\
S(x, x_0; t) &\approx \sum_{j=1}^{n} m_j \frac{(x_j - x_{0,j})^2}{2t} - \frac{1}{2}(V(x) + V(x_0))t.
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- Unfortunately: THEY ARE ALL INCORRECT, EVEN TO FIRST ORDER!!!!!

Example

Consider (for $n = 1$) the harmonic oscillator $H = \frac{1}{2m}(p^2 + m^2\omega^2 x^2)$. The exact formula for the action is

$$S = \frac{m\omega}{2\sin\omega t}\left[(x^2 + x_0^2)\cos\omega t - 2xx_0\right];$$

writing $\sin\omega t = \omega t + O(\Delta t^3)$, $\cos\omega t = 1 + O(t^2)$ we get

$$S = m\frac{(x - x_0)^2}{2t} - \frac{m\omega^2}{6}(x^2 + xx_0 + x_0^2)t + O(t^2)$$

while the mid-point rules above yield

$$S \approx m\frac{(x - x_0)^2}{2t} - \frac{1}{8}m\omega^2(x + x_0)^2 t$$

resp.

$$S \approx m\frac{(x - x_0)^2}{2t} - \frac{1}{2}m\omega^2(x^2 + x_0^2)t$$

(incompatible; they are not even consistent with each other...)

The correct formula is...

- ...obtained by making the Ansatz

$$S(x, x_0; t) = \sum_{j=1}^{n} m_j \frac{(x_j - x_{0,j})^2}{2t} + S_1(x, x_0; t)t$$

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- This gives (Makri and Miller (1988–89), de Gosson (2000)):

$$S(x, x_0; t) = \sum_{j=1}^{n} m_j \frac{(x_j - x_{0,j})^2}{2t} - \widetilde{V}(x, x_0)t + O(t^2)$$

where

$$\widetilde{V}(x, x_0) = \int_0^1 V(\lambda x + (1-\lambda)x_0, 0)d\lambda$$

is the average value of the potential (at time $t = 0$) along the line segment joining x_0 to x).

Short-time propagator

The short-time propagator is given by

$$\widetilde{G}(x,x_0;t)=\left(\frac{1}{2\pi i\hbar}\right)^{n/2}\sqrt{\widetilde{\rho}(t)}\exp\left(\frac{i}{\hbar}\widetilde{S}(x,x_0;t)\right)$$

where

$$\underbrace{\widetilde{S}(x,x_0;t)=\sum_{j=1}^{n}m_j\frac{(x_j-x_{0,j})^2}{2t}-\widetilde{V}(x,x_0)t}_{\text{Approximate action}}$$

and $\widetilde{\rho}$ is independent of x:

$$\widetilde{\rho}(t)=\underbrace{\frac{m_1\cdots m_n}{t^n}}_{\text{Approximate Van Vleck density}}.$$

Short-time propagator

The main property of the short-time propagator is that if

$$\Psi(x,t) = \int G(x,x_0;t)\Psi(x_0)d^n x_0$$

is the solution of Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi \quad , \quad \Psi(t=0) = \Psi_0$$

(G is the true propagator) then

$$\widetilde{\Psi}(x,t) = \int \widetilde{G}(x,x_0;t)\Psi_0(x_0)d^n x_0$$

satisfies

$$\Psi(x,t) - \widetilde{\Psi}(x,t) = O\left(t^2\right). \tag{1}$$

Consider a sharply located particle: $\Psi(x,0)=\Psi_0(x_0)=\delta(x-x_0)$. We have to specify its initial momentum: $p(0)=p_0$ (arbitrary!). We have to solve Bohm's equation

$$\dot{x}_j^{\Psi}=\frac{\hbar}{m_j}\operatorname{Im}\frac{1}{G}\frac{\partial\Psi}{\partial x_j}\Longleftrightarrow\dot{x}^{\Psi}=\hbar\operatorname{Im}M^{-1}\frac{\nabla_x G}{G}$$

where $M=\operatorname{diag}(m_1,...,m_n)$. Replacing G with its approximation $\widetilde{G}$ we get

$$\dot{x}^{\Psi}=\hbar\operatorname{Im}M^{-1}\frac{\nabla_x\widetilde{G}}{\widetilde{G}}+O(t^2).$$

Using formula

$$S(x,x_0;t)=\sum_{j=1}^{n}m_j\frac{(x_j-x_{0,j})^2}{2t}-\widetilde{V}(x,x_0))t+O(t^2)$$

yields the following equation:

$$\dot{x}^{\Psi}(t)=\frac{x^{\Psi}(t)-x^{\Psi}(0)}{t}-M^{-1}\nabla_x\widetilde{V}(x^{\Psi}(t),x_0))t+O(t^2).$$

This equation is singular at $t=0$ hence the initial condition $x^{\Psi}(0)=x_0$ is not sufficient to determine its solution. However, the additional condition $p(0)=p_0$ allows us to single out a trajectory: we have

$$x^{\Psi}(t)=x_0+\dot{x}^{\Psi}(0)t+O(t^2)=x_0+M^{-1}p_0t+O(t^2)$$

hence, inserting in the equation above, and using $\nabla_x\widetilde{V}(x_0,x_0)=\frac{1}{2}\nabla_xV(x_0,0)$,

$$\dot{x}^{\Psi}(t)=\frac{x^{\Psi}(t)-x^{\Psi}(0)}{t}-\frac{1}{2}M^{-1}\nabla_xV(x_0,0)t+O(t^2).$$

Differentiating with respect to t gives

$$M\ddot{x}^{\Psi}(t)=M\frac{x^{\Psi}(t)-x^{\Psi}(0)}{t^2}+M\frac{\dot{x}^{\Psi}(t)}{t}-\frac{1}{2}\nabla_xV(x_0,0)+O(t).$$

Equivalently

$$\dot{p}^{\Psi}(t) = M\ddot{x}^{\Psi}(t) = -\nabla_x V(x_0, 0) + O(t)$$

that is, integrating,

$$p^{\Psi}(t) = p_0 - \nabla_x V(x_0, 0)t + O(t^2).$$

Summarizing, the phase space motion is described by the two equations

$$\begin{aligned} x^{\Psi}(t) &= x_0 + M^{-1}p_0 t + O(t^2) \\ p^{\Psi}(t) &= p_0 - \nabla_x V(x_0, 0)t + O(t^2). \end{aligned}$$

These are the solutions, up to terms of order t^2, of the classical Hamilton equations:

$$\dot{x} = \nabla_p H(x, p, t) \quad , \quad \dot{p} = -\nabla_x H(x, p, t)$$

with initial values $x(0) = x_0$ and $p(0) = p_0$.

Quantum Zeno

This result provides a rigorous treatment of the “watched pot” effect: if we keep observing a particle that, if unwatched would make a transition from one quantum state to another, will now no longer make that transition. The unwatched transition occurs when the quantum potential grows to produce the transition. Continuously observing the particle does not allow the quantum potential to develop so the transition does not take place.

- I give the last word to Pablo Echenique-Robba (arXiv:1308.5619 [quant-ph]):

  "Shut up and let me think. Or why you should work on the foundations of quantum mechanics as much as you please"

This is a good advice, and this is why we are all here!

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- THANK YOU FOR YOUR KIND ATTENTION!