Non-commutative probability, conditional expectation values as weak values.

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Some Surprising Results.

1. von Neumann's 1931 approach is mathematically identical to Moyal's 1949

\[ \text{[von Neumann, Math. Ann. 104 (1931) 570-87]} \]

2. Moyal’s conditional expectation values of momentum and energy are intimately related to the energy-momentum tensor \( T^{\mu\nu}(x, t) \) of standard quantum field theory.

\[ \rho(x, t)P_M^j(x, t) = T^{0j}(x, t) \quad \text{and} \quad \rho(x, t)E_M(x, t) = T^{00}(x, t) \]


3. The Moyal momentum IS the Bohm momentum for the Schrödinger, Pauli and Dirac particles.

\[ \text{[Hiley, and Callaghan, Found. Phys. 42 (2012) 192-208]} \]

4. The Moyal/Bohm approach is about non-commutative probability theory.

\[ \text{[Hiley, arXiv 1211.2098]} \]

5. The Moyal/Bohm momentum is the weak value of the momentum operator,

\[ P_{\psi,B}^\mu(x, t) = \frac{\langle x|\hat{P}|\psi(x, t)\rangle}{\langle x|\psi(x, t)\rangle} \]

\[ \text{[Leavens, Found. Phys., 35 (2005) 469-91]} \]
\[ \text{[Hiley J. Phys.: Conference Series, 361 (2012) 012014.]} \]

6. Bohm momentum, energy, Bohm kinetic energy and hence the quantum potential can be measured using weak values.

\[ \text{[Rob Flack next lecture]} \]

7. Classical physics emerges from a non-commutative statistical (quantum) structure grounded in \textbf{process}.

\[ \text{[Hiley, Lecture Notes in Physics, vol. 813, pp. 705-750, Springer (2011).]} \]
The von Neumann 1931 Algebra.

Start with translations in \( x \) and \( p \).

\[
\hat{U}(\alpha) = \exp(i\alpha \hat{P}) \quad \text{and} \quad \hat{V}(\beta) = \exp(i\beta \hat{X})
\]

\([\hat{X}, \hat{P}] = i\]

Combine to give \( \hat{S}(\alpha, \beta) = \exp i(\alpha \hat{P} + \beta \hat{X}) \)

\(\alpha, \beta\) span a symplectic space.

\[\text{Symplectic Clifford algebra}\]

\[\text{E Enveloping algebra of Heisenberg algebra.}\]

\[\text{Non-commutative Probability}\]

\[\text{von Neumann shows}\]

\[\hat{A} \leftrightarrow a(\alpha, \beta) \quad \hat{A} = \int \int a(\alpha, \beta) \hat{S}(\alpha, \beta) d\alpha d\beta\]

\[\text{element of the symplectic space}\]

\[\text{‘symbol’}\]

Formally introduce \( \rho_\Psi(x) \to [ = |\psi\rangle\langle\psi| = \Psi_L(x)\Psi_R(x) ] \) and form \( F_\Psi(\alpha, \beta) = \text{Tr}[\hat{S}(\alpha, \beta)\rho_\Psi(x)]\)

\[\langle \hat{A} \rangle = \int \int a(\alpha, \beta) F_\Psi(\alpha, \beta) d\alpha d\beta\]

\[\text{looks like a probability measure?}\]

\[\text{Classical expectation value?}\]

Unfortunately \( F_\Psi(\alpha, \beta) \) can take negative values.

\[\text{Don’t worry we are in a non-commutative symplectic manifold.}\]

\[\text{Non-commutative Probability}\]


\[\text{[Groenewold, Physica, XII, (1946) 405-460].}\]
Non-commutative Phase Space.

Products of symbols

If $\hat{C} = \hat{A}\hat{B}$ then $C(x, x') = \int A(x, x'')B(x'', x')dx''$

von Neumann shows

$$c(\alpha, \beta) = a(\alpha, \beta) \star b(\alpha, \beta) = \int \int e^{2i(\gamma\beta - \delta\alpha)} a(\gamma - \alpha, \delta - \beta)b(\alpha, \beta)d\alpha d\beta$$

Moyal product

Special case:

$$\alpha \star \beta - \beta \star \alpha = i$$

Moyal chose new variables $\alpha \rightarrow x$ $\beta \rightarrow p$

Non-commutative Phase Space
A Closer look at the Non-commutative Moyal Algebra.

With star product we can form two types of bracket

\[
\{a, b\}_{MB} = \frac{a \star b - b \star a}{i\hbar} \quad \{a, b\}_{BB} = \frac{a \star b + b \star a}{2}
\]

Moyal bracket

Baker bracket

Moyal showed the star product can also be written in the form

\[
a(x, p) \star b(x, p) = a(x, p) \exp[i\hbar(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_x \vec{\partial}_p)/2]b(x, p)
\]

Easy to show that \( x \star p - p \star x = i\hbar \)

Then we have

\[
\{a, b\}_{MB} = 2a(x, p) \sin[\hbar(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_x \vec{\partial}_p)/2]b(x, p)
\]

\[
\{a, b\}_{BB} = a(x, p) \cos[\hbar(\vec{\partial}_x \vec{\partial}_p - \vec{\partial}_x \vec{\partial}_p)/2]b(x, p)
\]

The important property of these brackets is they contain the classical limit.

Moyal bracket becomes the Poisson bracket.

\[
\{a, b\}_{MB} = \{a, b\}_{PB} + O(\hbar^2) = [\partial_x a \partial_p b - \partial_p a \partial_x a]
\]

Baker bracket becomes a simple product

\[
\{a, b\}_{BB} = ab + O(\hbar^2)
\]
The Dynamics.

Because of non-commutativity

\[ H(x, p) \star F_\psi(x, p, t) = i(2\pi)^{-1} \int e^{-i\tau p} \psi^*(x - \tau/2) \overrightarrow{\partial}_t \psi(x + \tau/2) d\tau \]

\[ F_\psi(x, p, t) \star H(x, p) = -i \int e^{-i\tau p} \psi^*(x - \tau/2) \overleftarrow{\partial}_t \psi(x + \tau/2) \]

Subtracting gives Moyal bracket equation

\[ \partial_t F_\psi = (H \star F_\psi - F_\psi \star H)/2i = \{H, F_\psi\}_{MB} \]

Classical Liouville equation to \( O(\hbar^2) \)

\( \tau \rightarrow \hbar \tau \)

Adding gives Baker bracket equation \( \{H, F_\psi\}_{BB} = (H \star F_\psi + F_\psi \star H)/2 \)

\[ 2\{H, F\}_{BB} = i(2\pi)^{-1} \int e^{-i\tau p}[\psi^*(x - \tau/2) \overrightarrow{\partial}_t \psi(x + \tau/2) - \psi^*(x - \tau/2) \overleftarrow{\partial}_t \psi(x + \tau/2)] d\tau \]

Writing \( \psi = Re^{iS} \) we obtain

\[ \frac{\psi^* \overrightarrow{\partial}_t \psi}{\psi^* \psi} = \left[ \frac{\partial_t R(x + \tau/2)}{R(x + \tau/2)} - \frac{\partial_t R(x - \tau/2)}{R(x - \tau/2)} \right] + i \left[ \frac{\partial_t S(x + \tau/2)}{S(x + \tau/2)} + \frac{\partial_t S(x - \tau/2)}{S(x - \tau/2)} \right] \]

Go to the limit \( O(\hbar^2) \)

\[ H \star F_\psi + F_\psi \star H = -2(\partial_t S)F_\psi + O(\hbar^2) \Rightarrow 2(\partial_t S)F_\psi + \{H, F_\psi\}_{BB} = 0 \]

\[ \frac{\partial S}{\partial t} + H = 0 \]

Classical H-J equation.

No need for decoherence to reach the classical level
Time Development Equations.

\[ X - P \text{ Phase Space} \]

\[ \frac{\partial F}{\partial t} + [F, H]_{MB} = 0 \]

\[ 2 \frac{\partial S}{\partial t} F + [F, H]_{BB} = 0 \]

von Neumann/Moyal algebra
Moyal’s transport of $\psi$ is treated as a probability distribution.

What is the conditional expectation value of the momentum?

$$\rho(x)\bar{p} = \int pF_\psi(x, p)dp = \left(\frac{1}{2i}\right)[(\partial_{x_1} - \partial_{x_2})\psi(x_1)\psi(x_2)]_{x_1=x_2=x} \quad \text{Moyal momentum}$$

With $\psi = Re^{iS}$, we find

$$\bar{p}(x) = \frac{1}{2i}[\psi^*\nabla\psi - (\nabla\psi^*)\psi] = \nabla S \quad \text{Moyal momentum} = \text{Bohm momentum.}$$

Moyal’s transport of $\bar{p}$

$$\partial_t (\rho\bar{p}_k) + \sum_i \partial_{x_i} (\rho p_k \partial_{x_i} H) + \rho \partial_{x_k} H = 0$$

Again with $\psi = Re^{iS}$, we find

$$\frac{\partial}{\partial x_k} \left[ \frac{\partial S}{\partial t} + H - \frac{\nabla^2 \rho}{8m\rho} \right] = 0 \quad \text{Quantum potential.}$$

Or

$$\frac{\partial S}{\partial t} + H - \frac{\nabla^2 \rho}{8m\rho} = \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{1}{2m} \frac{\nabla^2 R}{R} = 0 \quad \text{Quantum Hamilton-Jacobi equation.}$$

For details see appendix of Moyal’s paper.


<table>
<thead>
<tr>
<th>$X - P$ Phase Space</th>
<th>Bohm Phase Space</th>
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<td>$\frac{\partial F}{\partial t} + [F, H]_{MB} = 0$</td>
<td>$\frac{\partial P_x}{\partial t} + \nabla_x \cdot \left( P_x \frac{\nabla_x S_x}{m} \right) = 0$</td>
<td>$p_B = \nabla_x S_x$</td>
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**Time Development Equations.**
Time Development of $\rho_\Psi(x,t)$ in Configuration Space.

For a pure state $\rho_\Psi(x,t)$ is idempotent and of rank one. In the symplectic non-commutative algebra we can form $\rho_\Psi = \Psi_L \Psi_R$ with $\Psi_L \in \mathcal{I}_L$ and $\Psi_R \in \mathcal{I}_R$.

Heuristic argument:

$|\psi\rangle\langle\psi| \Rightarrow \hat{\rho} = A\langle B\rightarrow A\epsilon B = \Psi_L \Psi_R = \rho_\Psi$

Standard ket $A\rangle$ Idempotent $\rangle\langle$

As before we have two time development equations

$$i\Psi_R(\overrightarrow{\partial}_t \Psi_L) = \Psi_R(\overrightarrow{H}\Psi_L)$$
and

$$-i(\Psi_R \overrightarrow{\partial}_t)\Psi_L = (\Psi_R \overrightarrow{H})\Psi_L$$

which we combine by adding and subtracting to find;

$$i \left[ (\overrightarrow{\partial}_t \Psi_L)\Psi_R + \Psi_L(\Psi_R \overrightarrow{\partial}_t) \right] = (\overrightarrow{H}\Psi_L)\Psi_R - \Psi_L(\Psi_R \overrightarrow{H}) = [H, \rho]_- \quad (A)$$

$$i \left[ (\overrightarrow{\partial}_t \Psi_L)\Psi_R - \Psi_L(\Psi_R \overrightarrow{\partial}_t) \right] = (\overrightarrow{H}\Psi_L)\Psi_R + \Psi_L(\Psi_R \overrightarrow{H}) = [H, \rho]_+ \quad (B)$$

From (A) we obtain $i\partial_t \rho = [H, \rho]_- \quad \text{Liouville equation}$

Conservation of Probability

From (B) we obtain $i\Psi_R \overrightarrow{\partial}_t \Psi_L = [H, \rho]_+ \quad \text{Conservation of Energy.}$

New equation?


### Time Development Equations.

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<td>$2 \frac{\partial S}{\partial t} \rho + [\rho, H]_- = 0$</td>
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- **von Neumann/Moyal algebra**
- **Bohm model**
- **Quantum algebra**
Project Quantum Algebraic Equations into a Representation.

Choose \( P_x = |x\rangle\langle x| \)

\[
\frac{\partial P_x}{\partial t} + \nabla_x \cdot \left( P_x \frac{\nabla_x S_x}{m} \right) = 0
\]

Conservation of probability

Quantum Hamilton-Jacobi equation.

Harmonic oscillator

Choose \( P_p = |p\rangle\langle p| \)

\[
\frac{\partial P_p}{\partial t} + \nabla_p \cdot \left( P_p \frac{\nabla_p S_p}{m} \right) = 0
\]

Possibility of Bohm model in momentum space.

But now \( x = -\left( \frac{\partial S_p}{\partial p} \right) \)

Trajectories from the streamlines of probability current.

\[
j_p = -\langle p | \frac{\partial}{\partial x} (\hat{V}(\hat{x})) | p \rangle
\]

What is called “Bohmian Mechanics” is but a fragment of the deeper non-commutative geometry.

But there is more!

Quantum potential

Eqs.

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2}
\]
Surprise Number 3:- Energy-Momentum Tensor.

\[ T^{\mu\nu} = - \left\{ \frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi)} \partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi^*)} \partial^\nu \psi^* \right\} \]

Take the Schrödinger Lagrangian:

\[ \mathcal{L} = -\frac{1}{2m} \nabla \psi^* \cdot \nabla \psi + \frac{i}{2} \left[ \psi^* (\partial_t \psi) - (\partial_t \psi^*) \psi \right] - V \psi^* \psi. \]

and find

\[ T^{0\mu} = -\frac{i}{2} \left[ (\partial^\mu \psi^*) \psi - \psi^* (\partial^\mu \psi) \right] = \frac{i}{2} \left[ \psi^* \overrightarrow{\partial} \cdot \psi \right] = -\rho \partial^\mu S \]

Recalling that

\[ P_M(x,t) = P_B(x,t) = \nabla S(x,t) \quad \text{and} \quad E_M(x,t) = E_B(x,t) = -\partial_t S(x,t) \]

Then explicitly:

\[ \rho(x,t) P^j_M(x,t) = T^{0j}(x,t) \quad \text{and} \quad \rho(x,t) E_M(x,t) = T^{00}(x,t) \]

These are the LOCAL expressions for the energy-momentum of the particle.

Conservation of energy is maintained through the quantum Hamilton-Jacobi equation.

Similar relations hold for the Pauli and Dirac particles. Use orthogonal Clifford algebra.


Standard QFT deals with the GLOBAL expression of energy-momentum

\[ P^j = \int T^{0j}(x,t) d^3 x \quad E = \int T^{00}(x,t) d^3 x \]
Surprise Number 4: Weak Values.

We will show that these quantities are related to weak values through:

\[ \rho P_{jB} = \rho \partial_j S = -T^{0j} = \Re[i \rho \langle P_j \rangle_W] \]

\[ \rho E_B = -\rho \partial_t S = -T^{00} = \Re[i \rho \langle P_t \rangle_W] \]

**Moyal/Bohm momentum.**

**Moyal/Bohm energy.**

What is a weak value?

\[ A_W = \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \]

N.B. \[ A_W \in \mathbb{C} \]


How do they appear in the formalism?

\[ \langle \psi | A | \psi \rangle = \sum \langle \psi | \phi_j \rangle \langle \phi_j | A | \psi \rangle \]

where \[ | \phi_j \rangle \] form a complete orthonormal set.

Then

\[ \langle \psi | A | \psi \rangle = \sum \langle \psi | \phi_j \rangle \left( \frac{\langle \phi_j | \psi \rangle}{\langle \phi_j | \psi \rangle} \right) \langle \phi_j | A | \psi \rangle = \sum \rho_j \frac{\langle \phi_j | A | \psi \rangle}{\langle \phi_j | \psi \rangle} \]

**Post select**

**Weak value.**

Remember \[ \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} \] is a complex number. It is clearly a transition probability amplitude.

But how is this related to the energy-momentum component \[ T^{0\mu}(x, t) \]?
Weak values when $\hat{P}$ is involved.

Weak value

$$\langle P \rangle_W = \frac{\langle x | P | \psi(t) \rangle}{\langle x | \psi(t) \rangle}$$

Form:

$$\langle x | \hat{P} | \psi(t) \rangle = \int \langle x | \hat{P} | x' \rangle \langle x' | \psi(t) \rangle dx' = -i \nabla \psi(x, t)$$

Write $\psi(x, t) = R(x, t)e^{iS(x, t)}$ then

$$\langle P \rangle_W = \nabla S(x, t) - i \nabla \rho(x, t)/2\rho(x, t) \quad \text{with} \quad \rho(x, t) = |\psi(x, t)|^2$$

Moyal/Bohm momentum. Osmotic momentum.

Real part of weak value:

$$\Re[i\rho \langle P \rangle_W] = [\nabla \psi^*(x)]\psi(x) - \psi^*(x)[\nabla \psi(x)] = \psi^*(x) \overset{\rightarrow}{\nabla} \psi(x) = \rho P_B \quad \text{Moyal/Bohm momentum} $$

$$T^{0j}(x, t)$$

Imaginary part of weak value:

$$\Im[-i\rho \langle P \rangle_W] = [\nabla \psi^*(x)]\psi(x) + \psi^*(x)[\nabla \psi(x)] = \nabla[\rho(x)].$$

The Bohm kinetic energy.

$$\Re[\langle P^2 \rangle_W] = (\nabla S(x))^2 - \frac{\nabla^2 R(x)}{R(x)} = P_B^2 + Q.$$

$$\Im[\langle P^2 \rangle_W] = \nabla^2 S(x) + \left(\frac{\nabla \rho(x)}{\rho(x)}\right) \nabla S(x).$$


Bohm Approach and Pauli spin.

Density element \( \rho(x) = \phi_L(x)\phi_R(x) \in \text{orthogonal Clifford algebra} \)

With \( \Psi = \begin{pmatrix} R_1e^{iS_1} \\ R_2e^{iS_2} \end{pmatrix} \)

The Bohm momentum and energy

\[
\rho P_B(x) = \rho_1(x) \nabla_x S_1(x) + \rho_2(x) \nabla_x S_2(x) = \Re[i\rho \langle P_j \rangle] \quad \text{Bohm Momentum}
\]

\[
\rho E_B(x) = \rho_1(x) \partial_t S_1(x) + \rho_2(x) \partial_t S_2(x) = \Re[i\rho \langle P_t \rangle] \quad \text{Bohm Energy}
\]

The Bohm kinetic energy is

\[
\Re[\langle P^2 \rangle] = P_B^2(x) + [2(\nabla_x W(x) \cdot S(x)) + W^2(x)] = P_B^2 + Q.
\]

Spin of particle \( S = i(\phi_L e_3 \tilde{\phi}_L) \) and \( \rho W = \nabla_x (\rho S) \) with \( \phi_R(x) = \tilde{\phi}_L \)


This generalises to the Dirac particle

Photon ‘trajectories’.

Experimental--Photons.

Problems with concept of a photon trajectory

We measure $T^{0j}(x, t)$, Poynting’s vector.

What is the meaning of the Poynting vector for a single photon?

What is the meaning of a photon at a point?

Must go to field theory

These criticisms do not apply to non-relativistic particles with finite rest mass (Schrödinger particle)

Need new experiments using atoms.