# Stochastic Mechanics of Relativistic Fields

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**configuration space:** $M$ manifold

**state space:** $TM$ tangent bundle

**dynamical variable:** function on $TM \times \mathbb{R}$

## Newtonian mechanics

**mass tensor:** $m_{ij}$

**kinetic energy:** $T = \frac{1}{2} m_{ij} v^i v^j$

Riemannian metric on $M$; $v^i$ velocity, $v_i$ momentum.

Take $M$ to be $\mathbb{R}^n$ with a flat metric.

**Newton's law:** $F_i = m_{ij} a^j$

**equations of motion:**

$$\dot{x}^i = v^i$$
$$\dot{v}^i = m^{ij} F_j$$

## Lagrangian mechanics

**Lagrangian:** dynamical variable $L = L(x, v, t)$

**path:** $X\colon \mathbb{R} \to M$ with velocity $\dot{X}$

**action:** $I = \int_{t_0}^{t_1} L(X, \dot{X}, t)\, dt$

**Hamilton's principle of least action:** $I$ stationary under variation of path

**Euler-Lagrange equation:** $\dfrac{\partial L}{\partial x^i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial v^i} = 0$

## Basic mechanics

$$\text{Basic} = \text{Newtonian} \cap \text{Lagrangian}$$

**potential energy:** $V$ defined by $L = T - V$

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x_i}$$

$$-\frac{\partial V}{\partial x^i} = \frac{d}{dt}\left(m_{ij}v^j - \frac{\partial V}{\partial v^i}\right)$$

$$\frac{d}{dt}(m_{ij}v^i) = F_i$$

$$F_i = -\frac{\partial V}{\partial x^i} + \frac{d}{dt}\frac{\partial V}{\partial v^i}$$

But $F_i$ is a dynamical variable, a function of position and velocity, so $\frac{\partial V}{\partial v^i}$ must be independent of the velocity. That is, the Lagrangian must be a

**basic Lagrangian:**

$$L = \frac{1}{2}m_{ij}v^i v^j - \varphi + A_i v^i$$

**scalar potential:** $\varphi$

**covector potential:** $A_i$

## Hamilton's principal function:

$$S(x,t) = -\int_t^{t_1} L(\big(X(s,x,t), \dot{X}(s,x,t), s\big)\, ds$$

A second form of the principle of least action is that $S$ be stationary when the flow is perturbed by a time-dependent vector field.

## Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \varphi = 0$$

For simplicity, we assume that the covector potential $A = 0$, so that $V = \varphi$ and

$$\frac{\partial S}{\partial t} + \frac{1}{2}\nabla^i S \nabla_i S + V = 0$$

## Basic stochasticization

Let $w$ be the Wiener process on $M$, the stochastic process of mean 0 characterized by

$$dw^i dw_i = \hbar dt + \mathrm{o}(dt)$$

We postulate that the motion of the configuration is a Markov process governed by the stochastic differential equation

$$dX^i = b^i\big(X(t), t\big)dt + dw^i$$

where $b^i$ is the forward velocity.

Thus the fluctuations are of order $dt^{\frac{1}{2}}$, and with a value larger than $\hbar$ this postulate could be falsified by experiment, without violating the Heisenberg uncertainty principle.

Now try to substitute the Markov process $X$ into Hamilton's principal function

$$S(x,t) = -\int_t^{t_1} L(\big(X(s,x,t), \dot{X}(s,x,t), s\big)\,ds$$

and require that the conditional expectation $\mathbb{E}_t$ given the configuration at time $t$ of the action be stationary with respect to variations of the forward velocity $b$.

The trajectories of the process $X$ are not differentiable, so replace the derivatives in $\dot{X}$ by difference quotients, and replace the integral by a Riemann sum. There is a singular term whose conditional expectation is 0, so it drops out, and there is a singular term that is a constant, independent of the trajectory, so it drops out from the variation. Then pass to the limit when the Riemann sum becomes an integral. The result is

**The stochastic principal function:**

$$S(x,t) = -\mathbb{E}_{x,t} \int_t^{t_1} \left( \frac{1}{2} b^i b_i + \frac{\hbar}{2} \nabla_i b^i - V \right) \big(X(s), s\big) ds$$

where $\mathbb{E}_{x,t}$ is the expectation conditioned by $X(t) = x$.

In addition to the forward velocity $b^i$ there are the

**backward velocity:** $b^i_*$

**current velocity:** $v^i = \dfrac{b^i + b^i_*}{2}$

**osmotic velocity:** $u^i = \dfrac{b^i - b^i_*}{2}$

The osmotic velocity depends only on the time-dependent probability density $\rho$. Let

$$R = \frac{\hbar}{2} \log \rho$$

Then

$$u^i = \frac{1}{\hbar} \nabla^i R$$

Computation shows that

$$\frac{\partial S}{\partial t} + \frac{1}{2}\nabla^i S \nabla_i S + V - \frac{1}{2}\nabla^i R \nabla_i R - \frac{\hbar}{2}\Delta R = 0$$
$$\frac{\partial R}{\partial t} + \nabla_i R \nabla^i S + \frac{\hbar}{2}\Delta S = 0$$

The first equation is the stochastic Hamilton-Jacobi equation. There is no deterministic analogue of the second equation since $R = 0$ when $\hbar = 0$. These two coupled nonlinear partial differential equations determine the process $X$.

With
$$\psi = e^{(R+iS)}$$
these equations are equivalent to the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\left(-\frac{1}{2}\Delta + V\right)\psi$$

This derivation is that of Guerra and Morato but using the classical Lagrangian. The result extends to the general case, when there is a covector potential $A_i$ and $M$ is not necessarily flat.

As I reported to this conference two years ago, there are problems with stochastic mechanics as a candidate for a physically realistic theory. There can be instantaneous signaling between widely separated correlated but dynamically uncoupled systems. Although the probability distribution of position measurements at a single time are the same for quantum mechanics and stochastic mechanics (because $|\psi|^2 = \rho$ is the probability density of the configuration), measurements at two different times, even when mutually compatible, can differ in the two theories.

The problem is that stochastic mechanics describes a local Markov process in multidimensional configuration space, not in physical space.

## Stochastic mechanics of fields

There are two motivations for applying stochastic mechanics to fields. One is the hope that since fields live on physical spacetime nonlocality problems may be avoided. The other is that it may provide useful technical tools in constructive quantum field theory.

The strategy is to apply basic stochasticization to a basic field Lagrangian.

Consider a real scalar field $\varphi$ on $d$-dimensional spacetime. I shall discuss the free field of mass $\mu > 0$ and the field with a $\varphi^4$ interaction.

Choose a spacelike hyperplane $\mathbb{R}^s$, where $s$, the number of space dimensions, is $d - 1$. The configuration space is a set of scalar functions $\varphi$ on $\mathbb{R}^s$.

Impose a spatial cutoff and a momentum cutoff. That is, put the system in a box of side $\lambda$ and represent the free field as a set of harmonic oscillators (a device going back to Jeans) with momentum bounded by $\kappa$. Then we have a system with finitely many degrees of freedom.

For the free field, of course, the limit

$$\lambda \to \infty, \kappa \to \infty$$

presents no difficulty. But it is unwise to attempt to formulate an interacting field on the same space since by Haag's theorem it must have an inequivalent representation of the canonical commutation relations.

Formally, the $\varphi^4$ theory has the interaction Hamiltonian

$$\int_{\mathbb{R}^s} \varphi^4(x)\, dx$$

With our cutoffs, the integral becomes a sum. But this expression is hopelessly singular as $\kappa \to \infty$, and the interaction Hamiltonian is replaced by

$$\int_{\mathbb{R}^s} :\!\varphi^4(x)\!:\, dx$$

The colons denote Wick ordering, which can be expressed simply by summing over quadruples of *distinct* oscillators.

For $d = 2$ it was shown that the Hamiltonian is bounded below by a constant independent of $\kappa$, giving the limit $\kappa \to \infty$. This was the origin of hypercontractivity, which has grown into a rich topic in analysis. The construction of the limit $\lambda \to \infty$ followed—in fact, with $\varphi^4$ replaced by any polynomial of even degree.

For $d = 3$ the $\varphi^4$ theory was rigorously established by Glimm and Jaffe, with mass and coupling constant renormalization, by heroic work with cluster expansions.

For $d > 4$ Aizenman established a no-go theorem: the limiting theory with cutoffs removed is non-interacting.

For $d = 4$ there are partial no-go results, and most workers in constructive quantum field theory believe that the limiting theory with cutoffs removed is non-interacting also for $d = 4$, but there is no complete proof—the problem is open.

In this work in constructive quantum field theory, Minkowski space was replaced by Euclidean space, giving a problem in probability theory: the Euclidean fields commute. Then the relativistic theory was reconstructed from the Euclidean theory.

This is a powerful analytic technique, but it does not give a candidate for quantum theory as emergent from an underlying classical theory. (But I heard or read somewhere that Stephen Hawking said something like "the world is Euclidean but appears relativistic." I don't have a reference for this or know what he meant if he did say something like this.)

What does stochastic mechanics have to say? We have a basic Lagrangian. But the separation of the energy for the free field into kinetic energy and potential energy is not a relativistically invariant procedure. Probability theory and relativity do not mix well, due to the indefinite nature of the Minkowski metric.

For the ground state of the free field, the corresponding random field of stochastic mechanics was constructed by Guerra and Ruggiero. It is a Gaussian field with Euclidean-invariant covariance function.

But it plays a different role than in Euclidean field theory; relativistic effects persist. Suppose that the field is coupled to an external potential localized in a bounded spacetime region $\Omega$, bounded in the past by $t_1$ and in the future by $t_2$. Consider the Markov process, not necessarily for the ground state, of the free field at time $t_1$. It interacts with the external potential and has a certain value at $t_2$. By the basic theorem of stochastic mechanics, the probability distribution of the process at time $t_2$ is the same as that given by quantum mechanics. Since the quantum field is relativistic, the influence of the external potential affects only the points at time $t_2$ that lie in the future cone of some point in $\Omega$.

But we cannot conclude a similar result for the Markov process itself. Superluminal influences persist, alas, and the hope that stochastic mechanics applied to field theory gives a reasonable candidate for quantum field theory as emergent from a classical theory does not seem to be valid. But it would be interesting to develop it further and see what it says about interacting fields.